Chapter 3 Symmetry and Group Theory

Chapter 3 deals with the fundamentals of the formal system used in this research, group theory. All basic constructs used in the research and based in group theory are presented here: group definitions, pictorial and discursive representations, graph representations, Cayley diagrams, group classifications, partial order lattices, isomorphisms, automorphisms, as well as permutations and combinatorics. All representations are given with respect to a singular structure, the symmetry group of the square, the dihedral group of order eight, to illustrate systematically the diverse aspects of the structure that each representation foregrounds.

3.1. Introduction

'The theory of groups is, as it were, the whole of mathematics stripped of its matter and reduced to pure form.' Poincare (1905)

'Numbers measure size; groups measure symmetry.' This first sentence of Armstrong's textbook 'Groups and Symmetry' (1988) is striking. Indeed the applications and the insights that group theory offers are many. From its first appearance disguised in the theory of equations to describe the effect of mapping of the different roots of a polynomial equation into themselves, to its various applications to number theory, combinatorics and especially to symmetry theory of geometrical figures there are many fascinating applications to explore. The focus here is the exploration of the application of group theory in symmetry theory. Symmetry of an object is a transformation that leaves the object unchanged. Formalizing this viewpoint requires a formulation of a mathematical characterization of symmetry. And still, the formalization is not enough; Klee asserts that the bilateral conformity of two parts, that is, the old definition of symmetry, has been superseded by the equalization of unequal but equivalent parts (Klee 1953). For Klee, the purely material balance of the scale finds its counter-part in the purely psychological balance of light and dark, weightless and heavy colors. Klee is right: It is the balancing and proportioning power of eye and brain that regulates the characterization of the object in terms of equilibrium and harmony. But all such entire world-making requires foundations. It is the premise of this work that all studies in formal composition should start from foundations and expand upon them. Group theory is a part of this foundation and it is argued here that it is a powerful tool that allows for possible re-descriptions in the analysis and description of an architecture work.

Here a very brief account of the history and logic of group theory is given to provide the foundations for the development of the model developed in this work. Formal accounts of group theory and in-depth analyses of its applications in the arts and particularly in the visual arts and architecture have been given in various sources and several of them are mentioned in this work below. The mathematical study of transformations, symmetry groups and abstract groups in general, has been given in various sources (Armstrong, 1988, Baglivo and Graver 1976, Budden 1978, Coxeter 1969, Coxeter and Moser 1972, Dorwart 1966, Grossman and Magnus 1964, Grünbaum and Shephard 1987, Jeger 1966, Lockwood and MacMillan 1978, March and Steadman 1971, Maxwell 1975, Shubnikov and Koptsik 1972, Toth 1964, Yaglom 1962, 1968, Yale 1968, Weyl 1952). The emphasis here is given to the representations and ways that group theory can be used to explain complexity in architectural design analysis and synthesis. A brief exposition of the formalism along with the formal tools that are used in the analysis and synthesis of form is given in the first part and a brief historical survey of the advance of group theory completes the chapter.

3.2. A first encounter

> *'If I am not mistaken, the word symmetry is used in our everyday language with two meanings. In the one sense symmetric means something like well-proportioned, and well-balanced, and symmetry denotes that sort of coincidence of several parts by which they integrate into a whole. Beauty is bound up with symmetry… The image of the balance provides a natural link to the second sense in which the word symmetry is used in modern times: bilateral symmetry, the symmetry of left and right, which is so conspicuous in the structure of the higher animals, especially the human body. Now this bilateral symmetry is strictly geometric and, in contrast to the vague notion of symmetry discussed before, an absolutely precise concept.' Weyl (1952)*

Group theory is the mathematical language of symmetry. There is no better way to understand the foundations and the premises of group theory than within one of its major applications in the study of geometrical figures. We take a square embedded in the plane *E* denoted by four vertices A, B, C, D. Two axes R_1 and R_2 are drawn perpendicular to the mid-edges of the square, and two more axes

R_3 and R_4 intersect at the point O, the center of the square. The square ABCD with the four axes and its center O are shown in Figure 3-1.

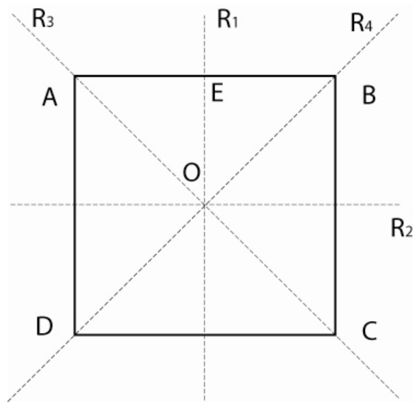


Figure 3-1: A square ABCD

Among all possible transformations of the plane *E* that contains the shape ABCD, there are eight that are quite special for the structure of the square ABCD. These transformations are: a reflection r_1 in the line R_1 bisecting the edges AB and CD; a reflection r_2 in the line R_2 bisecting the edges BC and DA; a reflection r_3 in the leading diagonal line R_3 connecting A and D; a reflection r_4 in the secondary diagonal line R_4; a rotation s_{90} by 90^0 clockwise about the center O the square; a rotation s_{180} by 180^0 clockwise about O; a rotation s_{270} by 270^0 clockwise about O; and a rotation s_0 by 0^0 or 360^0 clockwise about O, or more generally, a "do nothing" transformation, typically denoted by the symbol *e*. All these transformations are quite special in the way that when they operate on the square even if they move the individual vertices and edges of the square from one position to another, the overall shape appears unchanged. For example, in the case of the square ABCD, the transformation s_{90} puts A in the position occupied by B, B in the position occupied by C, C in the position occupied by D, and D in the position occupied by A. The transformation r_4 leaves A and C where they are and interchanges the points B and D respectively. Each of these eight transformations is called a 'symmetry of ABCD'. The collection of all these symmetry transformations that leave the structure of the square invariant is given in set notation below.

$$D_4 = \{e, s_{90}, s_{180}, s_{270}, r_1, r_2, r_3, r_4\}$$

An alternate representation of these transformations is suggested by the mapping of shapes upon these transformations. In this manner the eight transformations that leave the shape invariant may be represented as eight shapes that all together form the visual structure of the square. The square then is seen as an aggregation of eight lines that taken together form the structure of the square. The original line AB/2 is taken as the identity and all other parts are derived by the application of the transformations above. The collection of all these parts that form a square is given in set notation in Figure 3-2. All segments correspond one to one to the set of transformations above.

D_4 ={

}

Figure 3-2: Dihedral D_4 in set notation

This collection of transformations has remarkable properties. To begin with, they can all be combined using a rule as simple as the injunction "followed by". In this case, transformations are combined in series of any desired length to denote sequences of transformations the one following the other. There are alternative representational schemes that capture the conventions of such rules applications; here the symbol (*) is used to denote the rule "followed by" and the sequence of operations is meant to be read from right to left. For example a transformation $x*y$ means that a transformation y is followed by the transformation x. In the example of the square ABCD, a rule sequence such as r_1*s_{180} means that the rotation s_{180} by 180^0 clockwise about O is followed by the reflection r_1 in the line R_1. In this case the combined transformation r_1*s_{180} interchanges the points A and D as well as B and C. The result of the combined transformation is the same as the reflection r_2 in the line R_2. In this case we could write $r_1*s_{180} = r_2$. Any sequence of transformations that is produced by the combination of the eight transformations of the square will always produce one of the eight original transformations of the square. This is a property of this system that is formally called 'closure', that is, for any two elements x, y belonging in a set, their products, and in this case, the products $x*y$ and $y*x$, also belong in the group. But this collection of eight transformations has more exciting features that all nicely illustrate fundamental aspects of the theory of algebraic structures and more specifically of group theory.

First, there is a transformation that basically does nothing, the so-called identity transformation. This transformation in the example of Figure 3-1 is the transformation e. It is also clear that for each transformation in the set is another operation that can cancel it. In other words this means that for each transformation there is one that when it is combined with it produces the identity transformation. For example all mirror transformations when they are applied twice, bring the shape to its original position. Similarly two rotations by 180^0 clockwise about O bring the shape back to itself. Slightly more interesting but nevertheless obeying the same case are the rotations by 90^0 and 270^0 clockwise about O. It is easily seen that the rotation by 90^0 clockwise about O is the inverse of the rotation by 90^0 clockwise about O and the rotation by 270^0 is the inverse of the rotation by 90^0, both respectively clockwise about O. The identity is a bit more abstract in this but is essentially the same; the identity can be followed by itself and will have the shape stay stationery in its original position. This property of the system is the so-called *inverse* property. Finally, it is clear that for any three operations that are constructed as a series of the one following the other, the transformation $(x*y)*z$ is equivalent to doing z then y and x, as is doing the sequence z and y and then x, a sequence denoted as $x*(y*z)$. All these are remarkable properties that are all captured in the formal definitions of group theory. A succinct account follows below.

3.3. Group structure

A group is a set endowed with a rule; the set can be any collection and the elements of the set are whatever comprises this collection. The rule combines any ordered pair x, y of elements of the set and obtain a unique product xy which also lies in the set; from this definition it follows that both possible ways of combining any two elements, x, y, that is, xy and yx, also lie in the group. The rule is usually referred to as a multiplication or a composition on the given set.

A group is a set G together with a rule on G which satisfies three axioms: a) the multiplication is associative, that is to say, $(xy)z = x(yz)$ for any three, not necessarily distinct elements in G; b) there is an element e in G, called an identity element, such that $xe = x = ex$ for every $x \in G$; c) each element in G has an inverse x^{-1} which belongs to the set G and satisfies $x^{-1} x = e = x x^{-1}$. In general $xy \neq yx$ *(is different)*. However, when certain pairs of elements x, y in G obey $xy = yx$, it is said that these elements commute. The identity element e commutes with all elements of a group and every element commutes with its inverse. If all elements in G commute with each other, i.e., $xy = yx$ for all x, y of G, the group G is called commutative or Abelian.

A group is abstract if its elements are abstract, i.e., if they are not defined in any concrete way. A concrete example of an abstract group, i.e. a group with concrete elements with a law of composition, is called a realization of that abstract group. Such realizations might be groups of numbers, matrices, or geometric transformations. The structure of a group is the statement of the results of all possible compositions of pairs of elements. In general, the structure of a group can be defined in an analytical and a constructive way. The information about the structure of a group can be encoded descriptively in the explicit enumeration of all possible combinations of pairs of elements or constructively by a set of few elements and s set of rules that determine all possible combinations of pairs within the group. The analytical description of the structure of the group is given usually in a square array, the so-called multiplication table of the group, and the constructive description is given in a set of group generators and defining relations that apply on the generators.

3.3.1. Multiplication table

The explicit description of the structure of the group is given usually in a square array, the multiplication table of the group. This array provides the results of all combinations of the elements of the group. The multiplication table consists of columns and rows containing each a rearrangement of the elements of the group, and each entry in the square array denotes the product of the combination of the elements positioned in the corresponding outer rows and columns. This way of representation of the structure of the group was introduced in 1854 by Cayley and is very similar to the familiar multiplication tables of arithmetic. It follows from the definition of the multiplication table that for a finite group of order *n*, all possible binary combinations of the *n* elements are given by the formula n^2. These n^2 products are explicitly given in the multiplication table of the group. The concept of the multiplication table is illustrated here in Table 1 in the explicit illustration of the structure of the symmetry group of the square. The product x*y means first perform y then x; the products 82 of all four groups are computed by first taking the symmetry element in the top row and then combining it with the corresponding element in the left column.

Table 1.The multiplication table for the structure of the square

	e	s_{90}	s_{180}	s_{270}	r_1	r_2	r_3	r_4
e	e	s_{90}	s_{180}	s_{270}	r_1	r_2	r_3	r_4
s_{90}	s_{90}	s_{180}	s_{270}	e	r_4	r_1	r_2	r_3
s_{180}	s_{180}	s_{270}	e	s_{90}	r_3	r_4	r_1	r_2
s_{270}	s_{270}	e	s_{90}	s_{180}	r_2	r_3	r_4	r_1
r	r_1	r_2	r_3	r_4	e	s_{90}	s_{180}	s_{270}
r_2	r_2	r_3	r_4	r_1	s_{270}	e	s_{90}	s_{180}
r_3	r_3	r_4	r_1	r_2	s_{180}	s_{270}	e	s_{90}
r_4	r_4	r_1	r_2	r_3	s_{90}	s_{180}	s_{270}	e

All the properties of the group structure discussed so far are seen in a glance in this representation. More specifically, the representation of the groups in terms of the multiplication tables has three properties: a) each row and column contains each symbol exactly once; this property of the array corresponds to a fundamental theorem connecting groups and permutations; b) there is one row and one column that are identical with the top row and left column of the array respectively; this property of the array corresponds to the group axiom of the unit element or identity; c) the two entries *e* for the intersections of two elements in the array are symmetrically located with respect to the main diagonal; this property of the array corresponds to the group axiom on the existence of inverses.

3.3.2. Group generators

The constructive representation of a group describes groups independent of order and is given in a set of group generators and defining relations that apply on the generators. For an element x in a group G, by the axiom of closure, it follows that all powers of x, that is, x, xx, xxx, … or otherwise, x^1, x^2, x^3, … all belong in the group. Furthermore by the axiom of inverses the elements x^{-1}, $x^{-1}x^{-1}$, $x^{-1}x^{-1}x^{-1}$, … or otherwise x^{-1}, x^{-2}, x^{-3}, x^{-4},… all belong in the group too. x^0 is defined to be $x^0 = e$. If all elements of the group can be expressed as products involving only one element x and its inverse x^{-1}, then this element x is called a ‘generator of this group’ and the group is called a ‘cyclic group’. If $x^n = e$ for some n > 0, the least such n is called the ‘order of x’ and the element x is said to have a ‘finite order’. If no such n exists, x is said to have ‘infinite order’.

Similarly for two elements x and y in a group G then by the axiom of inverses, x and y are also in the group and so are $x^{-1}yx$, $xyx^{-1}y$, and so on. Any product that can be written using x and y as factors in any sequence and with any finite frequency is an element of the group and is called a ‘word’ (Baglivo and Graver, 1976). If all elements of the group can be expressed as products involving the elements x and y and their inverses x^{-1} and y^{-1}, then x and y are called the ‘generators of the group’ and the corresponding group is called a ‘dihedral group’. The concept of group generators can be extended to a set of more than two elements. If S is a set of elements of a group G and all elements of G can be expressed as products involving only the elements of S and their inverses, then the elements of S are the generators of G. Still, the generators are not enough to build by themselves the characteristics of the group; what is needed is an explicit definition of their relationships one with another. The second ingredient for the complete description of the structure of the group is a set of rules, that is, a set of defining relations that determine the structure of the group by group relations. If w is a non-empty word of group G such that $w = e$, then this equality is a relation of G. Since the word w is a product of generators of G, $w = e$ is a generating relation of G. The word w can be any product of generators and their inverses in any sequence and with any finite frequency. Two words w_1, w_2 that represent the same group element are equivalent words in a group G; equivalent words are in the same class and any of those can be taken as a representative of the class. The concepts of group generators and defining relations are used here to illustrate the symmetry structure of the square. The generators for the symmetry group of the square are the rotation of 90^0 about the center s and a reflection r about the mid-edge of the square. The defining relations are given by forming the words $w = e$ for each element separately and for the two together. The concept of group generators and defining relations can be easily generalized. A more intuitive description of these defining relations is given in the next section on the pictorial representation of the graph of the group. The relations for s, r and $s*r$ are given in Table 2.

Table 2: Generators and defining relations for the symmetry group of the square

Generators	s		Rotation of 90^0 about the center
	r		Reflection about an axis passing through the mid-edges
Relations	s4	=	e
	r2	=	e
	rsrs	=	e

3.3.3. Pictorial representation

A group can be defined as a network of directed segments specifying within its structure how any product of group elements corresponds to successive paths on the graph network. The representation of a group as a network of directed segments where the vertices correspond to elements and the segments to multiplication by group generators and their inverses, was invented by Cayley as well. Such a network or graph is often called a 'Cayley diagram'.

All graph networks of groups have in common certain fundamental properties (Baglivo and Graver, 1976). a) Every vertex of the graph is in one-to-one correspondence with a group element; b) Every edge of the graph is a directed segment and all edges with the same weight correspond to a ‘singular group generator’; c) Every path or sequence of directed segments within the graph corresponds to a word that represents a group element, and vice versa; d) Every succession of two paths within the graph corresponds to a composition or multiplication of two group elements; e) Any word for *e* corresponds to a closed path on the graph; and f) The graph of a group is a ‘connected network’; that is, there are paths from each vertex to every other vertex.

The Cayley diagrams are extremely useful tools in the examination of the structure of symmetry groups. The use of few rules, the so-called generators of the group, produce visually all the words that correspond to the vertices of the graph and characterize automatically the defining relations that control the structure of the groups; the complexity of the multiplicity of words is easily resolved within the structure of the graph. Significantly, all closed paths generate the defining relations for the group; closed paths involving a minimum number of steps or nodes are the defining relations mostly used. The sets of transformations that bring the shape into coincidence with itself can be nicely visualized as sets of directed segments or arrows that move or transform parts of the shape while the overall shapes remains invariant. The resulting graphs can be interpreted in perspectival projections, that is to say, seen as if the observer looks at the shapes through their front face in perspective so that the frontal face is closer and bigger and the back face is smaller and its edges are parallel to the front edges. The concept of the Cayley diagram is used to illustrate the symmetry structure of the square in Figure 3-3.

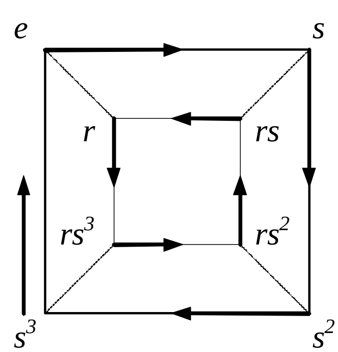


Figure 3-3: The Cayley diagram of the symmetry group of the square

3.3.4. Subgroups

One of the most interesting aspects of the structure of the group lies in its construal of its part (≤) relation. The elements of a symmetry group with an order *n* can form 2^n subsets. For example, in the case of the square there are $2^8 = 256$ possible subsets of symmetry transformations that leave the structure of the square invariant. Still very few of those have the very same properties with the symmetry group of the square that are part of. In general the subsets of the groups that form smaller groups within the big one are called subgroups and one of the most fascinating aspects of group theory is that the order of a finite group and the order of any of its subgroups are numerically related. This assertion is due to Lagrange back in 1771. Lagrange's theorem states that if H is a subgroup of a group G, and if the order of G is n, then the order m of H is a factor of n. In other words, the theorem specifies that the order of a finite group is a multiple of the order of any subgroup. From this, it follows that all prime-order groups have no proper subgroups.

There are several techniques that are used in the identification and generation of subgroups. In general, the enumeration of all subgroups for a given finite group is a very difficult task and has been carried through only for few groups; however, the generation of subgroups for groups of a small order is straight forward. The cyclic subgroups can be immediately picked out because every element of a group may be used to generate a cyclic subgroup. Given a group G and an element x of G, the set of all powers of x is a subgroup of G. This subgroup is called the subgroup generated by x and is written as $<x>$. If x has finite order m, then $<x> = \{e, x^1, x^2, x^3,... x^{m-1}\}$. If x has infinite order, then $<x>$ consists of infinite elements. In both cases the order of x is precisely the order of the subgroup generated by x. If there is an element x in G such that $<x> = G$, then G is a cyclic group.

Similarly, any subset of a group may be used to generate a subgroup. Given a group G and two elements x, y in a subset H of G, the set of all powers of x and y and their combinations is a subgroup of G. An expression of the form $x^m y^n$ for m, n any integers is a word in the elements of H. The collection of all these words is a subgroup of G. This subgroup is called the subgroup generated by H and written $<H>$. If there are elements x, y in H such that $<H> = G$, then the set H is a set for generators for G. The idea of a group generated by one or two elements may be extended for any number of generators.

Still, not every subset H is a subgroup. For example the subset $H = \{s_{90}, r_1, r_3\}$ fails because $s_{90} r_1 = r_3$, an element that does not belong in the set. The same set fails for many more reasons too, perhaps the most obvious being that it does not contain the identity. In all cases, for h and k lie in H then it should be: a) $h*k \in H$; b) $h^{-1} \in H$; and c) $I = h* h^{-1} \in H$. Table 3 displays the complete catalogue of the subgroups identified within the structure of the square. The symmetry group of the square has in all ten different subgroups.

Table 3: The ten subgroups of the symmetry group of the square

Subgroups	*Order of symmetry*
$\{e, s_{90}, s_{180}, s_{270}, r_1, r_2, r_3, r_4\}$	8
$\{e, s_{180}, r_1, r_2\}$	4
$\{e, s_{180}, r_3, r_4\}$	4
$\{e, s_{90}, s_{180}, s_{270}\}$	4
$\{e, r_1\}$	2
$\{e, r_2\}$	2
$\{e, r_3\}$	2
$\{e, r_4\}$	2
$\{e, s_{180}\}$	2
$\{e\}$	1

For a group G, a subgroup H of G, and an element $x \in G$, the set of elements $x*H$ defined by $x*H = \{x*h: h \in H\}$ is called a 'left coset' of H in G. The set of elements $H*x$ defined by $H*x = \{h*x: h \in H\}$ is called a 'right coset' of H in G. $x*H$ is the left coset of H containing (or generated by) x. $H*x$ is the right coset of H containing (or generated by) x. This rather abstract notion of the left and the right coset of a group can be nicely exemplified within the structure of the square. For the symmetry group of the square $D_4 = \{e, s_{90}, s_{180}, s_{270}, r_1, r_2, r_3, r_4\}$ and a subgroup $J = \{e, r_1\}$, for any element $x \in D_4$ the coset $J*x$ is defined to be:

$$\{e*x, r_1*x\}$$

We form the coset by multiplying each element of J by x and collecting the resulting elements into single sets. These sets are given in Table 4.

Table 4: The left cosets of the subgroup $J = \{e, r1\}$ for the symmetry group of the square

$J{*}e$	=	$\{e{*}e, r_1{*}e\}$	=	$\{e, r_1\}$
$J{*}s_{90}$	=	$\{e{*}s_{90}, r_1{*}s_{90}\}$	=	$\{s_{90}, r_4\}$
$J{*}s_{180}$	=	$\{e{*}s_{180}, r_1{*}s_{180}\}$	=	$\{s_{180}, r_2\}$
$J{*}s_{270}$	=	$\{e{*}s_{270}, r_1{*}s_{270}\}$	=	$\{s_{270}, r_3\}$
$J{*}r_1$	=	$\{e{*}r_1, r_1{*}r_1\}$	=	$\{r_1, e\}$
$J{*}r_2$	=	$\{e{*}r_2, r_1{*}r_2\}$	=	$\{r_2, s_{180}\}$
$J{*}r_3$	=	$\{e{*}r_3, r_1{*}r_3\}$	=	$\{r_3, s_{270}\}$
$J{*}r_4$	=	$\{e{*}r_4, r_1{*}r_4\}$	=	$\{r_4, s_{90}\}$

Several observations can be extracted from the computation in Table 4: a) the first four sets are equal to the last four sets forming thus only four distinct cosets; b) one of the four cosets is the subgroup *J* itself; c) no distinct cosets have any element in common; d) every element in D_4 lies in some coset; and e) each coset has the same number of elements. It is clear from observations (b) and (e) that each coset has two elements; from (a) and (c) it is deduced that the cosets when taken all together produce 4*2 = 8 elements which nicely explains that the order of *J* divides that of D_4 and that the result of doing the division will be the number of cosets.

Several other corollaries of these observations can be generalized and proved. These corollaries include: a) for a subgroup *H* of a group *G*, the *G* is a disjoint union of left cosets (or alternatively right cosets) of *H* in *G*; b) for a finite group *G* the order of an element of *G* divides the order of *G*; and c) every group of prime order is cyclic. All such observations are derived from the precise numerical relationship between groups and subgroups and the core of the Lagrange theorem. Here for example, using this theorem we know that if there are any subgroups in the symmetry group of the square that has an order of symmetry eight, these subgroups cannot have any orders other than 1, 2, 4, and 8. Still, this does not guarantee that these do exist. The only theorem we have for that is Sylow's theorem that proposes that if a number *m* is a power of a prime *k* and divides the order of a group *n*, then the group has a subgroup of order *m*. For the structure of the D_4 the possible orders of symmetry subgroups are 1, 2, 4, 8 . The numbers 2 and 4 are powers of a prime 2 and therefore there should be subgroups with such order. The ten subgroups of the symmetry group of the square are pictorially illustrated in Figure 3-4. Here each transformation tat leaves the square invariant is mapped as a line equal to AE equal to the half length of the side AB of the original square.

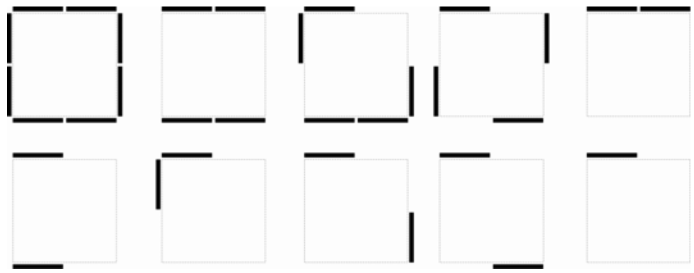


Figure 3-4: The ten subgroups of the symmetry group of the square

The same subgroup relation can be used to structure all possible symmetry subgroups of space. In fact all symmetry groups are subgroups of the Euclidean group *G* that consists of all possible isometries in the Euclidean space. All these symmetry groups are classified according to their translational structure and the dimensionality of the space that contains their elements (Yale, 1964). In Euclidean space there are ten symmetry groups *Gij*, for *i* = number of axes of translation and *j* = dimension of space, and $i \leq j$ and $j \leq 3$ (Economou (2001) as given in Table 5.

Table 5: The ten group structures of Euclidean space

	0-dimensional groups	*1-dimensional groups*	*2-dimensional groups*	*3-dimensional groups*
Point groups	G_{00}	G_{01}	G_{02}	G_{03}
Line groups		G_{11}	G_{12}	G_{13}
Plane groups			G_{22}	G_{23}
Space groups				G_{33}

The ten symmetry subgroups G_{ij} of the Euclidean space, for *i* the number of axes of translational symmetry and *j* the dimension of space, can further be decomposed in a series of subgroups according to the types of symmetry elements they contain and the ways these elements interact. There is one zero-dimensional point group G_{00}, one one-dimensional point group G_{01}, two one-dimensional line groups G_{11}, two two-dimensional point groups G_{02}, seven two-dimensional line groups G_{12}, seventeen two-dimensional plane groups G_{22}, fourteen three-dimensional point groups G_{03} which further split to seven finite polyhedral ones and seven infinite prismatic groups, nineteen three-dimensional line groups G_{13}, eighty three-dimensional plane groups G_{23}, and two-hundred thirty three-dimensional space groups G_{33}. The complete enumeration has been given in various sources E. Federov, A. Schonflies, and W. Barlow in the 1890s.

3.3.5. Lattices

The set of the symmetry subgroups of a particular symmetry group can be further sorted by a relation that orders all the subgroups in the set. If this relation can be established for all pairs of elements in the set then this relation is called 'total or strict order' and the set is called 'chain'. For instance, the relation "less than or equal to" (=) is a total order on integers, that is, for any two integers, one of them is less than or equal to the other. If this relation is defined for some, but not necessarily all, pairs of items, then the order is called 'partial order' and the set is called a 'partial ordered set' or 'poset'. For instance, the sets $\{x, y\}$ and $\{x, y, w\}$ are subsets of $\{x, y, z, w\}$, but neither is a subset of the other. In other words, the relation "subset" is a partial order on sets. Formally, both total order and partial order are relations that are reflexive, transitive and antisymmetric: Reflexive is a binary relation R for which aRa for all a. Transitive is that binary relation R for which aRb and bRc implies aRc. Antisymmetric is a binary relation R for which aRb and bRa implies $a = b$.

One of the most useful features of ordered sets is that, in the finite case, they can be drawn. Relationships between subsets of a set will be pictured in two ways (Dean 1970): a) 'Venn Diagrams' where each set is pictured as a subset of the plane and the subsets of interest are shaded; and b) 'Hasse Diagrams' where each set is designated by a dot on the plane and the R are designated with lines.

Typically graphs or Hasse diagrams are used to represent such order and show the nested relations of the subgroups diagrammatically in lattice diagrams. In graph representation, an empty relation between elements is represented by a graph with vertices and no lines connecting them and a complete relation between elements is represented by a graph with vertices that are all connected one to another. Different types of graphs represent types of hierarchies such as strict order, hierarchical order or semi-partial order. The graph of the symmetry group of the square is given in Figure 3-5. The diagram consists of four levels which correspond to the four possible orders of symmetry subgroups that are divisors of the maximum order of eight of the symmetry group of the square. The top level depicts the complete symmetry group of the square, the second and third levels depict the subgroups of orders four and two respectively, and the last level depicts just the subgroup that contains one element, the identity. All ten subgroups are positioned within the lattice.

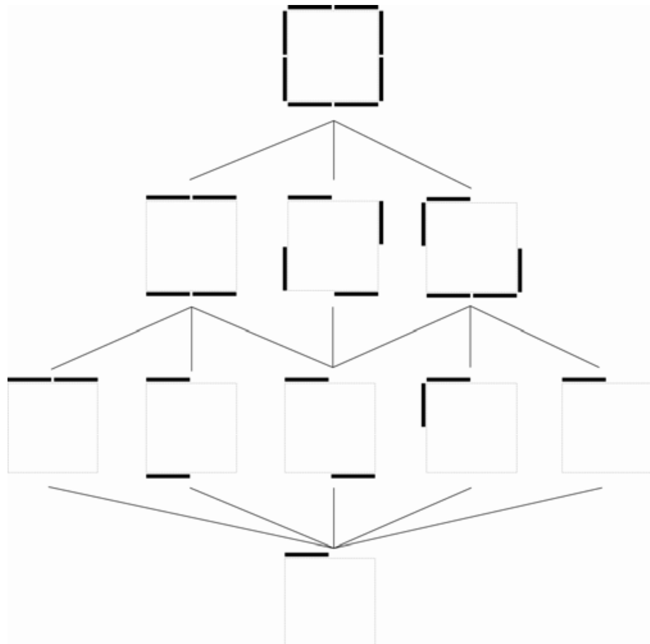


Figure 3-5: Order of subsymmetries of the ten subgroups of the symmetry group of the square

3.3.6. Conjugacy

The structure of the symmetry can be further worked out by sorting out the symmetry elements and the symmetry groups with other types of relations. A significant relation is one that partitions the sets of symmetry elements of a group into equivalent classes of isometries that are characterized by the same type, i.e., they impose the same type of transformation or rearrangement within a spatial structure. This relation is called 'conjugacy' relation and it is an equivalence relation between elements. For example, in the case of the square, a reflection through the horizontal mid-edges is similar to a reflection though the vertical mid-edge but both are different from the reflections that pass through the diagonals of the square. The choice of the spatial system within which the conjugacy relations are defined affects the ways that the designer looks at a design. Different conjugacy relations result in different decompositions of designs and different orderings of the resulting subgroups (March, 1996a, 1996b, 1996c).

Formally, given elements x, y, of a group G, x is conjugate to y if $g^{-1}xg = y$ for some $g \in G$. The equivalence classes are called 'conjugacy classes' and the elements within the same class must have the same order. The conjugacy class of an element x in G is found by calculating $g^{-1}xg$ for every $g \in G$. Similarly the conjugacy class of a power of x, say x^m, is found by calculating $g^{-1}x^mg$

for every $g \in G$. By algebra, once some of the conjugacy classes have been found, the other are easily calculated. Figure 3-6 shows three conjugacy classes for the symmetry subgroups of the symmetry group of the square. The panel a is the complete enumeration of all possible symmetry subgroups; the panel b considers reflections through edges and vertices as distinct and the panel c takes all reflections are typologically similar.

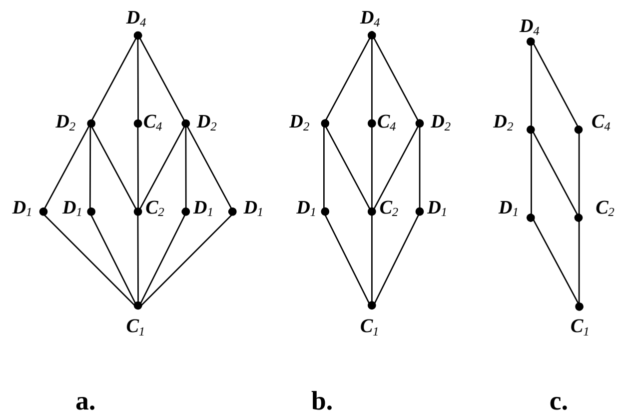


Figure 3-6: Partial order of the conjugacy classes of symmetry group of the square – a) 10 subgroups; b) 8 subgroups; c) 6 subgroups

3.3.7. Isomorphism

The structure of the group has been so far investigated though the products of the elements that comprise the groups and the relations order the resulting groups and subgroups. A very different direction of analysis is through the comparative analysis of diverse groups. One of the most interesting aspects of this new focus is the possibility of simplifying and the ability to recognize apparently different problems as basically the same. The relationship that allows it is called ‘isomorphism’. Formally, given two groups G and H, the groups are isomorphic if there is a bijection f: $G \rightarrow H$ such that for all $a, b \in G$ there is $f(a*b) = f(a)*f(b)$.

A specific class of bijections is quite interesting; the bijections between a set and itself are known as permutations of the set. A permutation is a rearrangement of objects. The collection of all permutations of K constructs a group S_k under composition of functions. If K consists of the first n positive integers, then S_k is written S_n and called the symmetric group of order n. The degree of S_n, that is the number of objects involved, is n. The order of S_n is n! and the elements of the permutation group are the n! rearrangements of the set. Permutations are nicely represented in the so-called ‘cycle notation’, whereas only the objects that are moved are written within a pair of brackets; in this notation a permutation *($x^1x^2...x^k$)* is called a cyclic permutation. The number *k* is its length and a cyclic permutation of length *k* is called a '*k*-cycle'; a *2*-cycle is usually referred as a 'transposition'. In section 3.3.1, Table 1 shows all the multiplications of functions for the 24 bijections that form a group.

Permutation groups or substitution groups are of particular interest because they provide concrete representations or realizations for all finite groups. Furthermore, permutations are important in the study of symmetry itself since any symmetry operation is a permutation of a set. A symmetry operation is not an arbitrary permutation but rather one that leaves the structure under consideration invariant; for example the symmetries of the square may be viewed as well as permutations of the four vertices that keep the structure of the square invariant. Similarly the symmetries of the Euclidean space are those permutations of the points in space that preserve distance.

Cayley's Theorem (Walker 1987) asserts that every group G is isomorphic to a subgroup of S_G, the group of all permutations of the set G. If G is finite and $|G| = n$, then S_G is isomorphic to S_n , the symmetric group of degree n. Any symmetry transformation can be considered as a specific permutation of a set that leaves the structure under consideration invariant. The decomposition of a two-dimensional shape in its basic spatial elements consisting of points and lines provides the best members for this set. For example, in the case of the square, the elements of the set that may be permuted by the symmetry transformations can be its four vertices or its four edges. Vertices and edges are not the only candidates for this set; in fact a description of a two-dimensional geometrical figure in terms of its internal diagonals is considered by mathematicians as the most elegant and parsimonious one because it has the simplest structure; it involves a minimum set of elements which completely specify the properties of the structure under consideration. The three descriptions of the structure of the square are given in Figure 3-7.

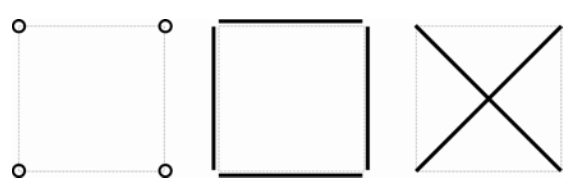


Figure 3-7: Description of the square in terms of its vertices, edges, and internal diagonals

The elements of the symmetry groups of the square can be written down in the form of cycles of permutations of a set consisting of the vertices, edges or diagonals of the square. The sum of all combinations or products of cycles of permutations under the elements of the symmetry group divided by the total number of the elements in the group, is the 'cycle index' of the corresponding permutation group.

The complete computation of the cycle index of the permutation group of the vertices is given here for the square and is illustrated in Figure 3-8. There are eight distinct transformations that bring the sqaure back to its original position and they are all isomorphic to eight permutations of vertices that leave the structure of the square invariant. The first transformation is the permutation corresponding to the rotation consisting of no motion at all. This permutation has four cycles of order one and is represented as f_1^4. The next permutation corresponds to a clockwise rotation of 90^0 around the center of the square and is comprised of one cycle of order four represented as f_4. The next permutation corresponds to a rotation of 180^0 around the center of the square and comprises the product of two cycles of order two, it is represented as f_2^2. The next permutation corresponds to a clockwise rotation of 270^0 around the center of the square and comprises one cycle of order four represented as f_4. The next two permutations correspond to the two reflections about the axes passing through the mid-edges of the square and each consists of a product of two cycles of order two represented as f_2^2. Finally there are two permutations that correspond to the two reflections about the vertices of the square and each is consists of a product of two cycles of order one and one cycle of order two represented as $f_1^2f_2$. The complete computation of the eight products of cycles of permutations is given in Figure 3-8.

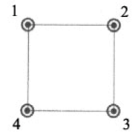
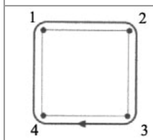
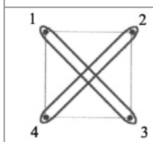
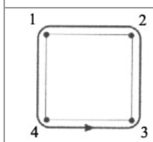
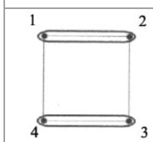
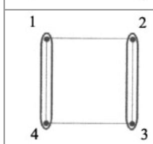
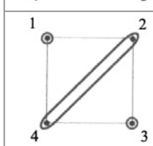
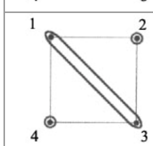
	→	$\begin{pmatrix}1&2&3&4\\1&2&3&4\end{pmatrix}$	→	(1) (2) (3) (4)	→	f_1^4
	→	$\begin{pmatrix}1&2&3&4\\2&3&4&1\end{pmatrix}$	→	(1 2 3 4)	→	f_4
	→	$\begin{pmatrix}1&2&3&4\\3&4&1&2\end{pmatrix}$	→	(1 3) (2 4)	→	f_2^2
	→	$\begin{pmatrix}1&2&3&4\\4&1&2&3\end{pmatrix}$	→	(1 2 3 4)	→	f_4
	→	$\begin{pmatrix}1&2&3&4\\2&1&4&3\end{pmatrix}$	→	(1 2) (3 4)	→	f_2^2
	→	$\begin{pmatrix}1&2&3&4\\4&3&2&1\end{pmatrix}$	→	(1 4) (2 3)	→	f_2^2
	→	$\begin{pmatrix}1&2&3&4\\1&4&3&2\end{pmatrix}$	→	(1) (3) (2 4)	→	$f_1^2 f_2$
	→	$\begin{pmatrix}1&2&3&4\\3&2&1&4\end{pmatrix}$	→	(1 3) (2) (4)	→	$f_1^2 f_2$

Figure 3-8: Complete visual computation of the cycle of permutation of vertices of the square

The sum of all compositions or products of the cycles of permutations under the elements of a symmetry group divided to the total number of the elements in the group is the cycle index of the corresponding permutation group of the vertices of the square. The cycle index C of the vertices of the square induced solely by rotations is given by the symbolic expression (3) and the complete index C of the vertices of the square is given by the symbolic expression (4).

$$C = f_1^{\,4} + 2f_4 + f_2^{\,2} \tag{1}$$

$$\underline{C} = f_1^{\,4} + 2f_4 + 3\, f_2^{\,2} + 2\, f_1^{\,2} f_2 \tag{2}$$

The relevance of the cycle index in general problems of combinatorics was first presented by *Polya* in his theorem of 'counting non-equivalent configurations' with respect to a given permutation group (Polya et al, 1983). March (2002) has emphasized the value of this theorem in a design context and in contemporary architectural research and Economou (1999) , (1999) provided several examples for the complete enumeration of non-equivalent configurations in cellular automata and sound structures have been given by Economou (1998, 1999).

3.3.8. Counting non-equivalent configurations

The relevance of the cycle index in general problems in combinatorics was first presented by *Polya* in his theorem of 'counting non-equivalent configurations' with respect to a given permutation group (Polya et al, 1983). Polya used to say 'The cycle index knows many things' (Polya et al, 1983, 67) and indeed this the case: Say that we wish to know in how many ways we can assign three colors or features on the vertices of a square, provided that we use one color per vertex, and we also want to count as distinct the 'enantiomorphs'; If we take the cycle index of the permutation group of the faces of the oblong induced by rotations, and a figure inventory $x+y+z$ for three colors x, y, and z, and substitute the figure inventory into the cycle index by replacing $f_k = x^k+y^k+z^k$, then, by expanding the cycle index in powers of x, y, and z, the resulting coefficient of, say, $x^r y^s z^t$ is the number of distinct ways we can paint r faces with a color x, s faces with a color y and t faces with a color z.

The appropriate method for this inquiry has been given by Polya in his theory on counting non - equivalent configurations with respect to a given permutation group (Polya et al, 1983). Essentially, Polya's theory of counting specifies the numbers of different ways we can assign k qualities to n vertices of an n-cornered figure without considering any two arrangements as different if they can be transformed one to another by a symmetry operation. In this case, the application of Polya's theorem specifies the different number of ways that $k = 1, 2, ...$ colors can be assigned in all n similar parts of the blocks. Note however that Polya's formalism provides the answer even if $k > n$; that means that in principle it is possible to know the different number of ways we can paint, say, the four vertices of a square, provided that we use one color per face, with any number of colors. The computation here limits the number of colors to two in order to present the computation in its simplest format. The figure inventory will be $x+y$. If we substitute the figure inventory into the two expressions (1) and (2) by replacing

$$f_k = x^k+y^k \quad [2] \tag{3}$$

The corresponding cycle indices of the rotational and the complete permutation groups of the vertices of the square are expanded in the expressions (3) and (4).

$$C_v = ((x+y)^4 + 2(x^4+y^4) + (x^2+y^2)^2)\ /4 \tag{4}$$

$$\underline{C}_v = ((x+y)^4 + 2(x^4+y^4) + 3(x^2+y^2)^2 + 2(x+y)^2(x^2+y^2))\ /8 \tag{5}$$

The expansion and computation of both symbolic sentences in (3) and (4) can be done with the binomial theorem given in (5).

$$(x+y)^n = \sum_{r+s=n} \frac{n!}{r!\,s!} x^r y^s \tag{6}$$

The details of the computation are not given here but are left to the interested reader. The result of the computation is given in (7).

$$\underline{C}_v = (8x^4 + 8x^3y + 16x^2y^2 + 8xy^3 + 8y^4)\ /8 = \\ x^4 + x^3y + 2x^2y^2 + xy^3 + y^4 \tag{7}$$

The coefficients of (7) give the numbers of non-equivalent configurations we can get using the structure of the square in a basic format of 2×2. The equation is symmetric with a vertical axis which means that results are the same for, say, the configuration of four white squares (x^4) and the four black quadrants (y^4). The computation also states that there should be two non-equivalent ways of arranging two white and two black quadrants (x^2y^2) upon the structure of the square. All non-equivalent configurations are given in Figure -3-9.

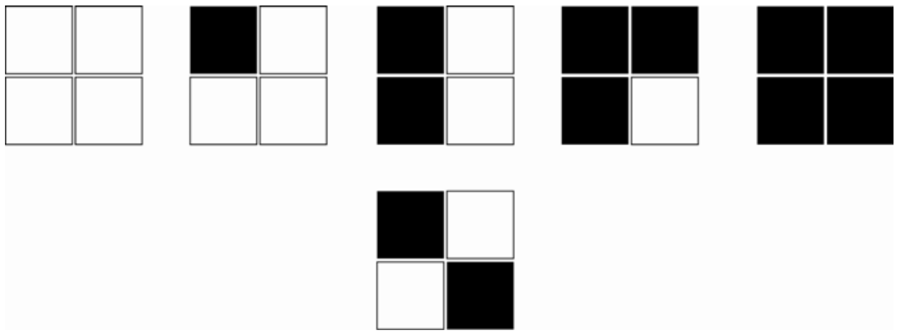


Figure -3-9: Non-equivalent configurations based on the symmetries of the square

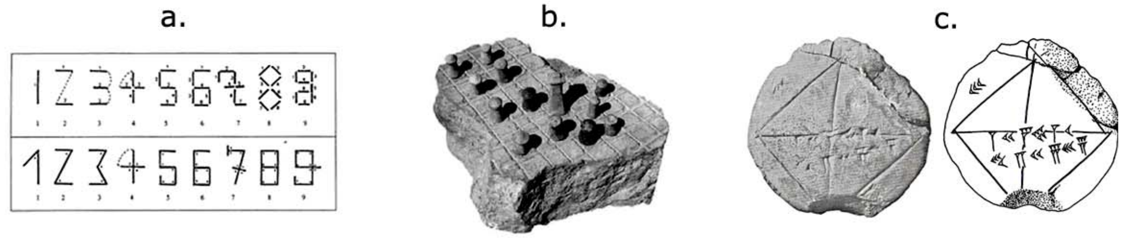
3.4. Tracing histories

'Do not be misled by the appearances. Things which look different may have the same meaning'. Al-Fullani (1732)

The history of the development of the fundamental mathematical concepts of group theory is an integral component of the development of mathematics in the late eighteenth and nineteenth century. The origins of the concept can be traced all the way back in the third millennium BCE to the symbolic systems found by the Sumerians and their studies in simple arithmetic (Neugebauer 1957). The addition table was the first act of abstraction that changed the meaning of addition. What the addition table does is assigning a definite number called a *sum* to every ordered pair of numbers. It would take several centuries of constant development and redevelopment of mathematical thought in various domains number theory and algebraic structures to eventually see addition as one of the earliest and most profound acts of mapping. A mapping is a function that assigns to each ordered pair of objects in a set another object from that set and the particular function encountered in the addition table of the Sumerians is called a binary operation. Similarly, Sumerians separated the operation of multiplication from its original meaning of finding the cardinal number of a rectangular array of objects by mapping ordered pairs of natural numbers into the systems of natural numbers. Two more gifts received from the Sumerians are the place value system, that is, the transposable meaning of a digit depending on its position in the written numeral, and the identity, what we call nowadays the zero element in addition and the unity element in multiplication (Neugebauer 1957). It would not be farfetched to fathom that Sumerians could represent numbers as points on a line, and therefore present an early species of the analytic geometry. However, the word number continued to refer to counting or measuring because of the isomorphism between cardinal and natural numbers. The true divorce will be marked later by the creation of number systems without numbers with the theory of sets to access a theoretical space. The task of the Cartesian thought was the shifting of this old paradigm towards analytical geometry. Out of the symbolic thought, it became clear that all of the knowledge of space and spatial relations could be translated into a new language of number system without numbers. Through these processes of translation and transformation, the true logical character of mathematical thought could be conceived in modern times.

Clearly, this history of the origins and development of group theory can never be claimed to be comprised of a unique trajectory; what really emerges through a sum of several trajectories is that each illuminates some aspects of the theory. Particularly interesting among them are diverse strands emerging in various parts of the non-western world, and occasionally peripheral in the compilation of this history. Some of these stands can be accounted as predecessors of group theory in non-western world and include cases such as: a) the numerals from the Sahara civilization transmitted to Europe through Spain in the tenth century CE (Smith and Ginsburg 1937); b) the oldest piece of chessboard found in Mohenjo-Daro, capital city of the Indus civilization (Canby 1961); c) the rational approximation of the diagonal of the square $\sqrt{2}$ in the eighteenth century BCE (Neugebauer 1957); d) the arithmetic formula of the truncated pyramid given in the Rhind papyrus (Banchoff 1990); e) the binomial expansion, known as the Pascal triangle shown in The writings by al-Maghribi in the twelth century CE (Ifrah 1985); f) the geometric resolution of cubic and quadratic equations by means of intersections of circles, parabolas and hyperbolas mentioned by Omar Khayyam in his Algebra book, in the twelth century CE (Sesiano 2000); g) tic-tac-toe arithmetical games practiced in Monomotapa, Africa by the seventh century CE (Zaslavsky 1973); h) geometric algorithms of the kind of Euler's Konisberg bridge manifested in the mukanda initiation rites in the Central African kingdoms since the fourteenth century CE (Gerdes 1999); i) magic squares practiced in the Islamic world since the tenth century CE – see 'Harmonious

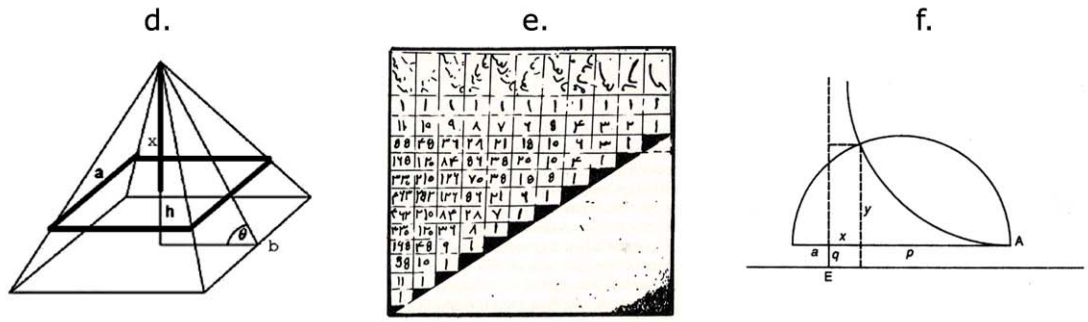
Dispositions of the Numbers' in al-Antaki's Book III, algorithmically defined by Al-Fullani al-Kishnawi al-Sudam, a native of Nigeria in the eighth century CE; and many others too.



Sahara-c.12000BC:
shape generation of number script
(Ifrah 1985)

Indus-c.3000BC:
the birthplace of chess game
(Canby 1961)

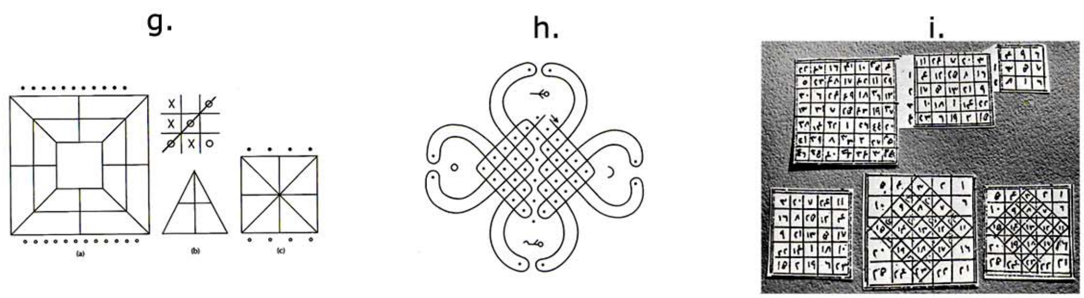
Summer-2000BC:
$(1;24,51,10)^2 = 1;59.59,59,381,40$
(Aaboe 1964)



Kmt: Ahmes - c.1800BC:
$b^3\text{-}a^3 = (b^2\text{+}ab\text{+}a^2)(b\text{-}a)$
(Banchoff 1990)

Morocco: al-Maghribi - c1100:
$(a + b)^n$
(Ifrah 1985)

Persia: Khayyam - c.1100
$x^3 + bx = ax^2 + c$
(Sesiano 2000)



Zimbabwe-c.600:
three-in-a-row or tic-tac-toe
(Zaslavsky 1973)

Angola-c.1400:
jokwe network of world genesis
(Zaslavsky 1973)

Nigeria: al-Fullani - 1732:
symmetries of the magic square
(Zaslavsky 1973)

Figure 3-10: Patterns exemplifying group theory applications in the non-western world

The core of group theory as it is understood nowadays was developed primarily during the eighteenth and nineteenth centuries out of the confluence of several and diverse investigations in fields of algebraic equations, permutations, number theory and others. Similarly the contemplation of the applicability of the emergent group theory to describe mathematical and physical aspects of space was also the product of these times. The beginning of the modern histories can track back to Monge's 'Descriptive geometry' (1794), the language of the engineer, an entirely new language that had as a task to shape objects by an exact measurement of their geometric properties. The work of Poncelet (1812-14), in his development of projective geometry, clearly built upon the work by Monge and constituted the foundations of the group theoretic approach in geometry, along with the descriptive geometry. This approach is particularly built around two issues: a) the elimination of the metric from geometry and b) the extension of the coordinate concept. The elimination of the metric information from geometry was achieved primarily by the dissociation of metric properties from the incidence properties. Poncelet, in his work on projective geometry, anticipated the analytical treatment of geometric figures, that is, the shift from synthetic projection to the analytical study of coordinate transformations in search of invariants and made it possible to apply invariant theory, rooted in number theory, to classify geometric objects. The extension of the coordinate concept was achieved by the shift of the meaning of coordinates from intervals to numbers. A coordinate system of a geometric manifold consists of independent parameters. Therefore, a space becomes a number manifold and this view of space separates the study of objective physical space from the study of mathematical spaces, and of physics from geometry.

The development of non-Euclidean geometries articulated even better this new vision of abstract, transformational geometries. The development of hyperbolic geometry independent of the parallel postulate of the Euclidean geometry ran into the epistemological problem of space. In order to separate geometry from physics, Riemann (1854) used the term 'space' to denote objective physical space and 'manifold' to denote mathematical space. The turn toward abstraction was completed with the introduction of *n*-dimensions. Gauss promoted the theory of algebraic equations and Lagrange in 1770 tried to determine why the solutions of cubic and quadratic equations work. Developing an approach of the combinatorial calculus type, he anticipated the subsequent permutation-based theory of solvability of algebraic equations. Cauchy by 1815 played a central role in shaping permutation theory. He elaborated the terminology for the concepts which we now call group, order of a group, index of a group, and subgroup. By such an arrangement, Cauchy meant an ordered string of quantities. A permutation or substitution denotes a transition from one arrangement to another (Wussing 1984). Galois in 1831 established that the algebraic equation $f(x) = 0$ of degree p^v is related to the structure of a group and that there is a connection between the solvability conditions of algebraic equations and permutation theory. By 1832 Galois reached the fundamental concept of the normality of certain subgroups but his work remained unknown until published fifteen years later by Liouville (1846). The Galois theory is regarded now as a 'show piece of mathematical unification, bringing together several different branches of the subject and creating a powerful machine for the study of problems of historical and mathematical importance' (Stewart 1992). The fundamental theorem of Galois theory establishes the correspondence between groups and fields.

The development of the concept of a permutation group marked the first stage in the evolution of the abstract group concept. Jordan's (1870) treatise must be regarded as crucial with its attempts to synthesize arithmetic and geometry by means of the permutation theoretical concept of a group. Closure under multiplication is declared then as the sole property required of a group (Wussing 1984). That is the case both in the definition of a group and in the presentation of Galois Theory. Cayley (1845) published his ideas on permutations and provided remarkable insight on the abstract conception of a group as a system of defining relations. The theory of invariants yielded the long-sought tool with which to bring to light connections between metric and projective geometry. Cayley (1859) used what is now known as the Cayley metric to embed Euclidean metric geometry in the general scheme of projective geometry, then came up with the concept of 'distance', defined as every relation that satisfies the condition

Dist. (P, P') + Dist. (P', P'') = Dist. (P, P'') for arbitrary positions of three points P, P', P''.

Because of his involvement with the determination of systems of invariants, Cayley did fail to discover the connection between the metrics and non-Euclidean geometries. At the time after Jordan's treatise was published, geometry came to be "a new attraction to the theory of permutations" (Wussing 1984). The decisive moments for the post-1870 evolution of the abstract group concept are those when the permutation-theoretic group concept invades geometry, leaving permutation theory behind.

The major catalyst for the unification of the various studies in group theory and its applicability as a fundamental construct for this unification of many and diverse geometries is really the Erlangen Program of 1872 by Klein. Klein in 1870 embarked on metrics associated with all types of quadratic curves and quartic surfaces and came up with plane and solid hyperbolic and elliptic geometries. Klein in 1871 wrote: "I wish to construct plane and space representations of the three geometries (Euclidean, Hyperbolic, and Elliptic) that would afford a complete overview of their characteristic features" (Klein 1921). Klein contributed to the formulation of the concept of group of transformations by forcing the transition to the explicit thinking in terms of groups. Logically and historically, there is a distinction between the use of group theoretic reasoning in geometry and the use of motions or transformations as group elements. The use of the group concept, in the form of group of transformations, for the purpose of classification in geometry brought a modified notion of motion in geometric thinking. The relation between physical motion and coordinate transformation shifted in favor of the physical view. This train of thought led to the idea that mathematics associated with motion must be pursued as the study of groups of motions and that the study of space could be facilitated by such as a framework of thought. Riemann and Helmholtz had attempted earlier to axiomatize geometry; if geometry is the structure of objective space, then it could be described in terms of the possible motions of physical bodies.

The linking of groups of geometric motions with their generators produced the advance that enabled Klein to apply the fundamental principles of permutation theory to geometry and to work out the concept of a discrete group of transformations. Furthermore the shaping of the abstract group concept by Cayley pointed towards an abstract view of groups, obviously influenced by the abstract position of Boole whose 'An Investigation of the Laws of Thought' was published in 1854. So, when Cayley came back to group theory in 1878 with his 'Theory of Groups' illustrated with the graphical representation of the groups, he stressed the role of generators, and received the long overdue recognition. The last pages of this initial theoretical grounding of group theory were written by Dyck (1882) and Burnside (1897). Dyck, one of Klein' students, took advantage of the parallel development of mathematical logic and wrote his 'Studies in Group Theory' that completed the elaboration of the abstract group concept. His concept of a group, remarkable for its historical objectivity, fulfilled all the requirements demanded of a fully developed abstract approach. Burnside's 'Theory of Groups of Finite Order' (1897) continued the study of group theory and prepared the ground for its applications in various fields and in particular in the study of symmetry.

3.5. Conclusions

The mathematical language of symmetry introduced in this chapter included all the fundamental concepts of group theory such as group axioms, elements, composition, associativity, identity, inverse, commutativity, structure, and order of a period of element. The analytical description of the structure of a group was given in terms of the concept and properties of a multiplication table and the constructive description of the structure of a group was given too in terms of the concepts of group generator, set of group generators, and sets of defining relations. The pictorial description of the structure of the group was presented including the concepts of the graph of group, Cayley diagram, directed network, correspondence of group element $\leftrightarrow$ graph vertex, group generator $\leftrightarrow$ graph directed weighted edge, group word $\leftrightarrow$ graph path, group composition of elements $\leftrightarrow$ graph succession of paths, and group identity word $\leftrightarrow$ graph closed path. The concept of subgroup was introduced including Lagrange's theorem and its relation to group generators and cosets. The concept of lattice representation of subgroups within groups was introduced including ordering relations such as strict order, hierarchic order, and partial order. The concept of conjugacy class was introduced along with equivalence class, conjugate elements, conjugate subgroups, and partial order of conjugacy classes. The idea of isomorphism was introduced with respect to the permutation groups the permutation groups and a basic account of Polya's theorem of counting non-equivalent configuration with respect to a given permutation group was given in the end.

The comprehensive review of group theory and symmetry presented above was structured around the square to provide a consistent set of parallel descriptions. It has helped to bring forward the logical framework upon which a significant body of research in architecture lies upon. Our path from the past to the present has opened up the issue of how a post-Cartesian shift on representation has dismantled old ways of see things. By developing a consistent treatment of the structure of the square through symbolic, theoretical and abstract representations, we have prepared the groundwork for the theoretical model developed in this research.

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